ON THE REFLECTION OF A PLANE SOUND WAVE FROM THE OPEN END OF A CIRCULAR TUBE

(OB OTRAZHENII PLOSKOI ZVUKOVOI VOLNY OT OTKRYTOGO KONTSA KRUGLOI TRUBY)

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Knowledge of the coefficient of reflection of a sound wave from the open end of a circular tube is essential for the study of oscillating combustion [1-3]. In the strict sense the problem of the diffraction of sound waves at the open end of a circular tube with rigid walls without a flange has been solved in [4-7]. There, it was assumed that the medium outside and inside the tube is homogeneous and at rest. In problems related to combustion the gas is moving. The medium outside the tube should not be considered homogeneous since combustion products that escape from the tube may considerably differ in their properties from the gas surrounding the tube. One may select a case where the flow velocity is small and has no influence upon the radiation of sound from the tube, whereas the difference of the thermodynamic parameters of the gas in the stream and the surrounding space does significantly influence the magnitude of the reflection coefficient. An attempt to calculate the effect of nonhomogeneity of the medium was undertaken in [8]. However, because of the error that was contained there in the writing of the boundary conditions it is required that this problem be studied again. This paper presents a derivation of the formula for the coefficient of reflection of a plane sound wave from the open end of a circular semi-infinite tube with absolutely rigid walls considering a contact discontinuity at the boundary between the stream and the surrounding medium.

The circular tube of radius a (Fig. 1) with absolutely rigid, infinitely thin walls is semi-infinite and has no flange. Its axis coincides with the z-axis of a cylindrical coordinate system r, z and extends along $z < 0$. Axial symmetry is assumed.

We study steady vibrations, and the time dependence of the parameters of the acoustic field is described by a function of the form $\exp(-i\omega t)$.

Assuming the gas outside and inside the tube to be at rest we can write the equations of the acoustic field in the form

$$\triangle\psi_j + k_j^2\psi_j = 0 \qquad \left(j = 1, 2;\ k_j^2 = \frac{\omega^2}{c_j^2};\ \triangle = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{\partial^2}{\partial z^2}\right) \qquad (1)$$

Here c_j is the speed of sound. The indices 1 and 2 refer to the parameters of the medium in the regions $r < a$ and $r > a$, respectively. The velocity potential ψ_j is related to the acoustic pressure and velocity by the relations

$$p_j = i\omega\rho_{0j}\psi_j, \qquad \mathbf{v}_j = \nabla\psi_j \qquad (\rho_{0j} \text{ is the density of the medium}) \qquad (2)$$

The potential ψ_j should satisfy the following boundary conditions

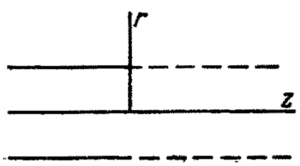

Fig. 1.

$$\frac{\partial\psi_j}{\partial r} = 0 \quad \text{at} \quad r = a,\ z < 0 \qquad (3)$$

$$\frac{\partial\psi_1}{\partial r} = \frac{\partial\psi_2}{\partial r} \quad \text{at} \quad r = a,\ z > 0 \qquad (4)$$

$$\rho_{01}\psi_1 = \rho_{02}\psi_2 \quad \text{at} \quad r = a,\ z > 0 \qquad (5)$$

A plane sound wave of amplitude A impinges from the left on the mouth of the tube. A diverging wave is created outside the tube, and a plane wave of amplitude B is reflected inside the tube. The ratio $R = B/A$ becomes the unknown coefficient of reflection of a plane wave from the mouth of a tube. We shall assume that $k_1 a < 3.832$; then the plane mode will be a unique mode of propagation [9,10]. Consequently

$$\psi_1 \sim Ae^{ik_1z} + Be^{-ik_1z} \quad \text{as} \quad z \to -\infty \qquad (6)$$

At sufficiently large distances from the mouth of the tube one can write ψ_2 as

$$\psi_2 \sim f(\theta)\frac{e^{ik_2R}}{R} \quad \text{at} \quad R = \sqrt{r^2 + z^2} \to \infty \qquad (7)$$

Here θ is at a given point the angle between the wave front and the z-axis.

Let us introduce into the study the following functions

$$h(z) = \frac{1}{\rho_{01}}[\rho_{02}\psi_2(a, z) - \rho_{01}\psi_1(a, z)], \qquad w(z) = \left.\frac{\partial\psi_1}{\partial r}\right|_{r=a} = \left.\frac{\partial\psi_2}{\partial z}\right|_{r=a} \qquad (8)$$

According to (3) and (5)

$$h(z) \neq 0, \quad w(z) = 0 \quad \text{at} \quad z < 0; \qquad h(z) = 0, \quad w(z) \neq 0 \quad \text{at} \quad z > 0 \qquad (9)$$

After carrying out a Fourier transformation and denoting the transforms by capital letters we obtain

$$\Psi_j(\zeta, r) = \int_{-\infty}^{\infty} \psi_j(z, r)\, e^{-i\zeta z} dz \tag{10}$$

Instead of (1) we now have

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{d\Psi_j}{dr}\right) + (k_j^2 - \zeta^2)\Psi_j = 0 \qquad (j = 1, 2) \tag{11}$$

The solutions of Equations (11), which satisfy the condition of emission at infinity and of finiteness at the z-axis, become

$$\Psi_1(\zeta, r) = C(\zeta) J_0(r\sqrt{k_1^2 - \zeta^2}), \qquad \Psi_2(\zeta, r) = D(\zeta) H_0^{(1)}(r\sqrt{k_2^2 - \zeta^2}) \tag{12}$$

Here J_n is an nth order Bessel function of real argument, $H_n^{(1)}$ is an nth order Hankel function of the first kind, $C(\zeta)$ and $D(\zeta)$ are arbitrary functions.

From (8) and (12) it follows that

$$H(\zeta) = \frac{\rho_{02}}{\rho_{01}} D(\zeta) H_0^{(1)}(a\sqrt{k_2^2 - \zeta^2}) - C(\zeta) J_0(a\sqrt{k_1^2 - \zeta^2}) \tag{13}$$

$$W(\zeta) = -\sqrt{k_1^2 - \zeta^2}\, C(\zeta) J_1(a\sqrt{k_1^2 - \zeta^2}) = -\sqrt{k_2^2 - \zeta^2}\, D(\zeta) H_1^{(1)}(a\sqrt{k_2^2 - \zeta^2}) \tag{14}$$

Eliminating the arbitrary functions $C(\zeta)$ and $D(\zeta)$ from (13) and (14) we obtain

$$z_1^2 L(\zeta) H(\zeta) = \frac{2}{a} W(\zeta) \tag{15}$$

$$L(\zeta) = \frac{2\rho_{01} J_1(az_1) H_1^{(1)}(az_2)\, z_2 / z_1}{\rho_{01} a z_2 J_0(az_1) H_1^{(1)}(az_2) - \rho_{02} a z_1 J_1(az_1) H_0^{(1)}(az_2)} \qquad (z_j = \sqrt{k_j^2 - \zeta^2}) \tag{16}$$

Note that in the limiting case of $k_1 = k_2$ and $\rho_{01} = \rho_{02}$, Equation (15) goes over to Equation (V.2) of [7].

It is more convenient to continue the analysis if it is assume that the constants k_j are complex numbers and that they have small imaginary parts Im $k_j > 0$.

We shall show that

$$R = \frac{\mathrm{res}_{\zeta=-k_1} H(\zeta)}{\mathrm{res}_{\zeta=k_1} H(\zeta)} \tag{17}$$

From the asymptotic behavior of the function ψ_1 as $z \to \infty$ it follows that

$$H(\zeta) = -\frac{iA}{\zeta - k_1} - \frac{iB}{\zeta + k_1} + \varphi(\zeta) \tag{18}$$

Here $\varphi(\zeta)$ is a regular function in the region $\mathrm{Im}\ \zeta > -\mathrm{Im}\ k_1$, and consequently, Formula (17) is correct. Thus, the problem reduces to the finding of the function $H(\zeta)$ from Equation (15). Let us study this. Equation (15) makes sense only when its left and right sides have a joint region of analyticity in the complex plane of the variable ζ. The study of the properties of the functions $H(\zeta)$ and $W(\zeta)$ does not differ from that carried out in [7]. It is evident from (18) that the function $z_1^2 H(\zeta)$ is analytic in the half-plane $\mathrm{Im}\ \zeta > -\mathrm{Im}\ k_1$. Inasmuch as the asymptotic behavior of the function $w(z)$ as $z \to \infty$ allows the inclusion of a factor of the form $\exp(-ik_j z)$ one can assert that the function $W(\zeta)$ is analytic in the region $\mathrm{Im}\ \zeta < \mathrm{Im}\ k_j$. The region of analyticity of the function $L(\zeta)$, whose explicit form is given by Formula (16), is the strip $|\mathrm{Im}\ \zeta| < \mathrm{Min}\ (\mathrm{Im}\ k_j)$, since the points $\zeta = \pm k_j$, where the singularities occur, lie outside this strip.

For future reference, it is also essential to note that the function $L(\zeta)$ does not go to zero in this strip. Actually, in accordance with the previously made assumption, the unique propagating mode in the tube is a plane wave. This means that the points $\zeta = \pm i(\alpha_{1n}^2 - k_1^2)^{1/2}$, where $J_1(\alpha_{1n} a) = 0$, have a finite imaginary part, and consequently, lie outside the strip of regularity of the function $L(\zeta)$.

The function $H_1^{(1)}(az_2)$ does not have any zeros in the region $-\pi/2 \leqslant \arg z_2 \leqslant 3\pi/2$, [11]. Choosing one branch of the function $z_2(\zeta)$ we find that the argument of this function lies within the limits $|\arg z_2| \leqslant \pi/2$ in the studied strip. Consequently, the function $H_1^{(1)}(az_2)$ does not have any zeros which lie within the strip.

The studied properties of Equation (15) allow us to find its solution with the aid of the Wiener-Hopf technique [12]. Using the fact that a function which is analytic in the strip can be represented in the form of a sum of two functions, one of which is analytic in the upper half-plane starting from the lower edge of the strip, and the other is analytic in the lower half-plane starting from the upper edge of the strip, we can represent $L(\zeta)$ in the form

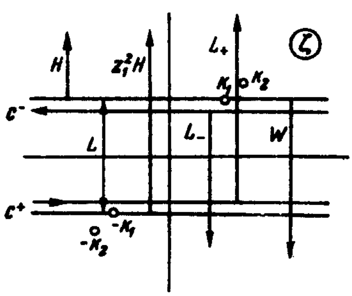

Fig. 2.

$$L(\zeta) = \frac{L_+(\zeta)}{L_-(\zeta)} \tag{19}$$

where
$$L_+(\zeta) = \exp\left[\frac{1}{2\pi i}\int\limits_{C+}\frac{\ln L(t)\,dt}{t-\zeta}\right]$$

$$L_-(\zeta) = \exp\left[-\frac{1}{2\pi i}\int\limits_{C-}\frac{\ln L(t)\,dt}{t-\zeta}\right] \tag{20}$$

The contours of integration C^+ and C^- and also the regions of regularity of all studied functions are shown in Fig. 2. In the final result one should assume $\operatorname{Im} k_j = 0$. When $\operatorname{Im} k_j \to 0$ the contours C^+ and C^- coincide with the real axis with the exception of the singular points of the integrand. The way of circumventing the latter is clear from Fig. 2.

Using (19) one can write Equation (15) as follows

$$z_1^2 L_+(\zeta) H(\zeta) = \frac{2}{a} L_-(\zeta) W(\zeta) \tag{21}$$

Here the left-hand side of Equation (21) is regular in the half-plane $\operatorname{Im} \zeta > \operatorname{Max}(-\operatorname{Im} k_j)$ while the right-hand side is regular in the region $\operatorname{Im} \zeta < \operatorname{Min}(\operatorname{Im} k_j)$

Thus Equation (21) describes an integral function in the ζ-plane.

The evaluation of the asymptotic behavior of the functions $H(\zeta)$ and $W(\zeta)$ for large ζ is carried out just as in [7], starting from physical considerations of the behavior of the velocity potential and its derivative as $z \to 0^{\pm}$. One can obtain

$$H(\zeta) \sim (-i\zeta)^{-\alpha-1}, \qquad \alpha > 0 \quad \text{for } |\zeta| \to \infty, \operatorname{Im} \zeta > 0 \tag{22}$$

$$W(\zeta) \sim (i\zeta)^{\beta-1}, \qquad \beta < 1 \quad \text{for } |\zeta| \to \infty, \operatorname{Im} \zeta < 0 \tag{23}$$

The evaluation of the asymptotic behavior of $L_+(\zeta)$ for large ζ can be accomplished using the asymptotic behavior of cylindrical functions for large values of their arguments and representing $L_+(\zeta)$ in the following form

$$L_+(\zeta) = \exp\left[\frac{\zeta}{\pi i} \int_0^\infty \frac{\ln L(t)\, dt}{t^2 - \zeta^2}\right] \tag{24}$$

When we break up the path of integration into the segments $(0, k_2)$ and (k_2, ∞) and take into account the uniform convergence of the integrals, we can show that

$$L_+(\zeta) \sim (-i\zeta)^{-1/2} \quad \text{for } |\zeta| \to \infty, \operatorname{Im} \zeta > 0 \tag{25}$$

Similarly

$$L_-(\zeta) \sim (i\zeta)^{1/2} \quad \text{for } |\zeta| \to \infty, \operatorname{Im} \zeta < 0 \tag{26}$$

If we substitute the asymptotic estimates (22), (23), (25) and (26) into Equation (21) we find by means of the Liouville theorem that both parts of (21) are equal to a constant. Consequently,

$$H(\zeta) = \frac{\text{const}}{(k_1^2 - \zeta^2) L_+(\zeta)} \tag{27}$$

It follows from (24) that

$$L_{+}(-k_1) = \frac{1}{L_{+}(k_1)} \tag{28}$$

With the aid of (17), (27) and (28) we find that

$$R = |R|\, e^{2i\delta} = -[L_{+}(k_1)]^2 \tag{29}$$

Using the explicit form of the function $L_{+}(k_1)$ we obtain an expression for the modulus and the phase of the reflection coefficient

$$|R| = \exp\left\{\frac{2k_1}{\pi}\int_0^{k_2} \tan^{-1}\left[-\frac{J_1(az_2)}{N_1(az_2)}\right]dt - \right.$$

$$-\frac{2k_1}{\pi}\int_0^{k_1} \tan^{-1}\left[-\frac{z_2 J_0(az_1)\, J_1(az_2) - \rho_{21} z_1 J_1(az_1)\, J_0(az_2)}{z_2 J_0(az_1)\, N_1(az_2) - \rho_{21} z_1 J_1(az_1)\, N_0(az_2)}\right]\frac{dt}{t^2 - k_1^2} -$$

$$\left. -\frac{2k_1}{\pi}\int_{k_1}^{k_2} \tan^{-1}\left[-\frac{z_2 I_0(az_1^*)\, J_1(az_2) + \rho_{21} z_1^* I_1(az_1^*)\, J_0(az_2)}{z_2 I_0(az_1^*)\, N_1(az_2) + \rho_{21} z_1^* I_1(az_1^*)\, N_0(az_2)}\right]\frac{dt}{t^2 - k_1^2}\right\} \tag{30}$$

$$2\delta = \frac{2k_1}{\pi}\sum_{i=1}^{5} F_i + \pi \tag{31}$$

Here

$$\rho_{21} = \rho_{02} / \rho_{01}, \quad z_j^* = \sqrt{t^2 - k_j^2}$$

$$F_1 = -\int_0^{k_1} \ln\left[\frac{z_2}{z_1} J_1(az_1)\sqrt{J_1^2(az_2) + N_1^2(az_2)}\right]\frac{dt}{t^2 - k_1^2}$$

$$F_2 = -\int_{k_1}^{k_2} \ln\left[\frac{z_2}{z_1^*} I_1(az_1^*)\sqrt{J_1^2(az_2) + N_1^2(az_2)}\right]\frac{dt}{t^2 - k_1^2}$$

$$F_3 = \int_0^{k_2} \ln\{[z_2 a J_0(az_1)\, J_1(az_2) - \rho_{21} z_1 a J_1(az_1)\, J_0(az_2)]^2 +$$

$$+ [az_2 J_0(az_1)\, N_1(az_2) - \rho_{21} a z_1 J_1(az_1)\, N_0(az_2)]^2\}^{1/2}\frac{dt}{t^2 - k_1^2}$$

$$F_4 = \int_{k_1}^{k_2} \ln\{[az_2 I_0(az_1^*)\, J_1(az_2) + \rho_{21} a z_1^* I_1(az_1^*)\, J_0(az_2)]^2 +$$

$$+ [az_2 I_0(az_1^*)\, N_1(az_2) + \rho_{21} a z_1^* I_1(az_1^*)\, N_0(az_2)]^2\}^{1/2}\frac{dt}{t^2 - k_1^2}$$

$$F_5 = -\int_{k_2}^{\infty} \ln\left\{\frac{I_1(az_1^*)\, K_1(az_2^*)\, z_2^* / z_1^*}{az_2^* I_0(az_1^*)\, K_1(az_2^*) + \rho_{21} a z_1^* I_1(az_1^*)\, K_0(az_2^*)}\right\}\frac{dt}{t^2 - k_1^2}$$

Here J_n is an nth order Bessel function of real argument, N_n is an nth order Neumann function, I_n is an nth order Bessel function of imaginary argument, and K_n is an nth order MacDonald function.

Formulas (30) and (31) are true if $k_2 > k_1$. It is not difficult to obtain the corresponding expressions for $k_2 < k_1$.

When the vibration frequency ω and the radius of the tube a are such that $k_j a \ll 1$, one can use in Formula (30) the series representations of the cylindrical functions and retain only small terms of first order.

Then, after integration we find

$$|R| = \exp\left[-\frac{k_1 k_2 a^2}{2}\frac{\rho_{02}}{\rho_{01}}\right] \tag{32}$$

Formula (32) holds for $k_2 > k_1$.

When the space is occupied by a homogeneous medium it follows from (32) that

$$|R| = \exp[-k^2a^2/2], \quad \text{or} \quad |R| = 1 - k^2a^2/2$$

which is in agreement with [7,9,10].

It is interesting to note that according to (32) in the case of a heated stream and a cool surrounding gas, when $c_1/\gamma_1 > c_2/\gamma_2$, the separation from the open end of the tube is larger (the reflection coefficient is smaller) than the separation in a homogeneous medium, regardless of whether this medium is characterized by the parameters of 1 or 2.

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